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COMMON FIXED POINT THEOREM USING IMPLICIT RELATION

Rajesh Shrivastava*[1], Arihant Jain[2] & Amit Kumar Gupta[3]

[1]Department of Mathematics, Govt. Shyama Prasad Mukherji Science & Commerce College, Bhopal (M.P.) India

[2]School of Studies in Mathematics, Vikram University, Ujjain (M.P.) India

[3]Department of Mathematics, Govt. Shyama Prasad Mukherji Science & Commerce College, Bhopal (M.P.) India

ABSTRACT

The object of this paper is to use the concept of compatible and reciprocal continuity of mappings and prove fixed point theorems for eight such mappings satisfying an implicit relation. We have also cited an example in support of our result.

Keywords: *Common fixed point, compatible maps, weakly compatible maps, reciprocals continuity of maps, complete metric space.*

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I. INTRODUCTION

After the classical results of Jungck [3] of common fixed point of two commuting mappings, Sessa [13] initiated the weaker condition than that of commutativity namely weak commutativity of maps and proved the results regarding fixed point consideration of such maps. Of course two commuting mappings are weakly commuting but the converse is not true always. Further a weakly condition of these notions namely, compatibility of maps has been introduced by Jungck [4]. He has proved the result regarding common fixed point of such maps. Jungck [4] also demonstrated that commuting mappings were weakly commuting and weakly commuting were compatible but neither implications were reversible.

After the introduction of compatibility, various types of compatibility namely compatibility of type (A) introduced by Jungck et. al. [6], compatible mappings of types (B) introduced by Pathak et. al. [8], compatibility of type (C) introduced by Pathak et. al. [10], compatibility of type (P) introduced by Pathak et. al. [9].

Recently Jungck and Rhoades [7] has introduced a more weaker class among all commutative conditions namely weakly compatibility or coincidently commutative of maps and gave results regarding common fixed points of such maps. Pant [11] introduced the notion of reciprocal continuity of maps in such a way that continuity implies reciprocally continuity of maps, but the converse is not always true.

In a paper, Popa [12] by using the notion of compatibility, weakly compatibility and reciprocal continuity of maps presented a general fixed point theorem for four such maps satisfying an implicit relation and extend the result for six maps.

In this paper, we have made appropriate corrections and then we have extended the result of Popa [12] by taking eight mappings as opposed to six mappings using the improved implicit relations given by Bouhadjera & Djoudi [1] and proved some common fixed point theorem by using the notion of compatibility, weakly compatibility and reciprocally continuity of mappings in complete metric space. Our results of eight mappings are seen to be probably new and unreported in the literature which opens a wider scope. To demonstrate the validity of the hypothesis a related example has also been furnished.

II. PRELIMINARIES

Thought this paper (X, d) stands for metric space.

Definition 2.1.[13] Two self maps S and T of a metrics space X are said to be weakly commutative if

$$d(STx, TSx) \le d(Sx, Tx), \quad \forall\, x \in X.$$

Definition 2.2. [5] Two self maps S and T of a metric space X are said to be compatible if
$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Sx_n = Tx_n = t$, for some t in X.

Definition 2.3. [7] Two self maps S and T of a metric space X are said to be weakly compatible if they commute at coincidence points. i.e. Ax = Bx for some x in X, then
ABx = BAx.
It is easy to see that two compatible maps are weakly compatible but converse is not true

Definition 2.4. [11] Two self maps S and T of a metric space X are said to be reciprocal continuous if $\lim_{n\to\infty} TSx_n = Tt$ and $\lim_{n\to\infty} STx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some t in X.

Definition 2.5. [7] Two self mappings A and B of a metric space X are said to be occasionally weakly compatible(owc) iff there is a point x in X which is coincidence point of A and B at which A and B commute.

Proposition 2.1. [4, 5, 6, 7, 8, 9, 10, 11, 12]

1. Commutativity implies weak commutativity but converse is not true.
2. Weak commutativity implies compatibility but converse is not true always.

Implicit relation : [12]

Let Φ be a set of real functions $F(t_1,\dots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions :

(F_1): F is non-increasing in variable t_5 and t_6

(F_2): there exists $h \in (0, 1)$, such that for every with
(F_a): $F(u, v, v, u, u + v, 0) \le 0$ or (F_b) $F(u, v, u, v, 0, u + v) \le 0$ we have $u \le hv$.

(F_3): $F(u, v, v, u. u + v, 0) > 0$, for all $u > 0$

Example 2.1. [12] $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{ t_2, t_3, t_4, ½ (t_5+t_6)\}$, where $k \in (0, 1)$

Example 2.2. [12] $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - t_1 (at_2 + bt_3 + ct_4) - dt_5t_6$,
where $a > 0, b, c, d \ge 0$, $(a + b + c) < 1$ and $(a + d) < 1$.

Example 2.3. [12] $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - k \max\{t_1 - k \max\{t_2^2, t_3t_4, t_5t_6\}\}$, where $k \in (0, 1)$.

Example 2.4. [12] $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - t_1 [at_2^p + bt_3^p + ct_4^p]^{1/p} - d\sqrt{t_5t_6}$,
where $0 < a < (1-d)^p$, $b, c, d, \ge 0$, $a + b + c < 1$ and $d < 1$, $p \in N^*$.

Example 2.5. [12] $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - c(t_3^2t_4^2 + t_5^2t_5^2)/ (1 + t_2 + t_3 + t_4)$, where $c \in (0, 1)$.

III. MAIN RESULT

Theorem 3.1. Let A, B, S, T, P, Q, R and I be self-mappings of a complete metric space (X, d) such that following conditions are satisfy -

(3.1.1) $AB(X) \subset RI(X)$ and $ST(X) \subset PQ(X)$

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(3.1.2) for all x, y $\in$ X with $d(PQx, ABx) + d(RIy, STy) \neq 0$,

$$F\{d(ABx,STy),d(PQx,RIy),d(PQx,ABx),d(RIy,STy),d(PQx,STy),d(RIy,ABx)\} \leq 0,$$

where F $\in\Phi$

Or

(3.1.3) $d(ABx, STy) = 0$, If $d(PQx, ABx) + d(RIx, STy) = 0$.

(3.1.4)(a) If (AB, PQ) is a compatible pair of reciprocally continuous mappings and (ST, RI) is occasionally weakly compatible pair of mappings.

Or

(3.1.4)(b) (ST, RI) is a compatible pair of reciprocally continuous mapping and (AB, PQ) is occasionally weakly compatible pair of mapping

Then, AB, ST, PQ and RI have a unique common fixed point say z.

(3.1.5) If the pair (A, B), (A, PQ), (B, PQ), (S, T), (S, RI) and (T, RI) commute at z. then A, B, S, T, PQ and RI have a unique common fixed point. Furthermore if

(3.1.6): the pair (P, Q), (P, AB), (Q, AB), (R, AB), (R, PQ), (I, AB) and (I, PQ) commute at z then A, B, S, T, P, Q, R and I have a unique common fixed point.

Proof. Suppose x_o be an arbitrary point in X. since $AB(X) \subset RI(X)$, we can find a point $x_1 \in X$ such that $ABx_o = RIx_1$. Also since $ST(X) \subset PQ(X)$, we can further choose a point $x_2 \in X$ such that $STx_1 = PQx_2$. Inductively we can construct sequences.

$$\{x_n\} \text{ and } \{y_n\} \text{ by } y_{2n} = ABx_{2n} = RIx_{2n+1}$$

And

$$y_{2n+1} = STx_{2n+1} = PQx_{2n+2}, \quad \text{for } n = 0, 1, 2\ldots\ldots$$

Case I: If $d(PQx_{2n}, ABx_{2n}) + d(RIx_{2n+1}, STx_{2n+1}) \neq 0$

Using (3.1.2), we have successively

$$F\{d(ABx_{2n},STx_{2n+1}), d(PQx_{2n},RIx_{2n+1}), d(PQx_{2n}, ABx_{2n}), d(RIx_{2n+1}, STx_{2n+1}), d(PQx_{2n}, STx_{2n+1}), d(RIx_{2n+1}, ABx_{2n})\} \leq 0$$

Or

$$F\{d(ABx_{2n},STx_{2n+1}), d(STx_{2n-1}, ABx_{2n}), d(STx_{2n-1}, ABx_{2n}), d(ABx_{2n},STx_{2n+1}), d(STx_{2n-1}, STx_{2n+1}), d(ABx_{2n}, ABx_{2n})\} \leq 0$$

or

$$F\{d(ABx_{2n},STx_{2n+1}), d(STx_{2n-1}, ABx_{2n}), d(STx_{2n-1}, ABx_{2n}), d(ABx_{2n},STx_{2n+1}), d(STx_{2n-1}, ABx_{2n}) + d(ABx_{2n},STx_{2n+1}),0)\}\leq 0.$$

Using (F_a), we have

$$d(ABx_{2n},STx_{2n+1}) \leq h d(ABx_{2n},STx_{2n-1}). \qquad (i)$$

Similarly if

$$d(PQx_{2n}, ABx_{2n}) + d(RIx_{2n-1}, STx_{2n-1}) \neq 0.$$

Using (3.1.2), we have

$$F\{d(ABx_{2n},STx_{2n-1}), d(PQx_{2n},RIx_{2n-1}), d(PQx_{2n}, ABx_{2n}), d(RIx_{2n-1}, STx_{2n-1}), d(PQx_{2n}, STx_{2n-1}), d(RIx_{2n-1}, ABx_{2n})\} \leq 0$$

or $$F\{d(ABx_{2n},STx_{2n-1}), d(STx_{2n}, ABx_{2n-2}), d(STx_{2n-1}, ABx_{2n}), d(ABx_{2n-2},STx_{2n-1}), d(STx_{2n-1}, STx_{2n-1}), d(ABx_{2n-2}, ABx_{2n})\} \leq 0.$$

Using (F_b), we have

$$d(ABx_{2n},STx_{2n-1}) \leq hd(STx_{2n-1}, ABx_{2n-2}) = hd(ABx_{2n-2},STx_{2n-1}).$$

Thus by (i), we have

$$d(ABx_{2n},STx_{2n+1}) \leq h^2 d(ABx_{2n-2},STx_{2n-1}).$$

Continuing this process, we get

$$d(ABx_{2n},STx_{2n+1}) \leq h^{2n} d(ABx_o,STx_1).$$

Now it can be easily seen that the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, therefore there exist a point z in X such that $\lim_{n\to\infty} y_n = z$.

Moreover

$$y_{2n} = ABx_{2n} = RIx_{2n+1} \to z \quad \text{and } y_{2n+1} = STx_{2n+1} = PQx_{2n+2} \to z.$$

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Suppose that (AB, PQ) is a compatible pair of reciprocally continuous mappings, we have

$$(AB)(PQ)x_{2n} \longrightarrow ABz\ ,\quad (PQ)(AB)x_{2n} \longrightarrow PQz$$

and

$$\lim_{n\to\infty} d\{(AB)(PQ)x_{2n}, (PQ)(AB)x_{2n}\} = 0$$

which gives $d(ABz, PQz) = 0$ i.e. $ABz = PQz$.
Since $AB(X) \subset (RI)(X)$, therefore there exists a point w in X such that $ABz = RIw$.
thus, $ABz = RIw = PQz$
Using (3.1.2), we have
$F\{d(ABz,STw), d(PQz,RIw), d(PQ_{z,} ABz), d(RIw, STw), d(PQz, STw), d(RIw, ABz)\} \le 0$ Or
$F\{d(ABz,STw), 0, 0, d(ABz,STw), 0, d(ABz,STw), 0\} \le 0$
Now from (F_a), we have $d(ABz,STw) \le h.0$ or $d(ABz,STw) \le 0$
implies that $ABz = STw$.
Hence, $ABz = RIw = STw = PQz$
Since the pair (AB, PQ) compatible and hence occasionally weakly compatible yield that $(AB)(PQ)z = (PQ)(AB)z$ and $(AB)(AB)z = (AB)(PQ)(z = (PQ)(AB)z = (PQ)(PQ)z$.
By weak compatibility of (ST, RI), we have

$$(ST)(ST)w = (ST)(RI)w = (RI)(ST)w = (RI)(RI)w.$$

Since, $d\{(PQ)(AB)z, d((AB)(AB)z) + d(RIw, STw)\} = 0.$
Then from (3.1.3) it follows that

$$d\{(AB)(AB)z, (ST)w\} = 0 \text{ or } d\{(AB)(AB)z, (AB)z\} = 0$$

implies that $(AB)z = (AB)(AB)z$
or $(AB)z = (AB)(AB)z = (PQ)(AB)z$.
Hence, (AB)z is common fixed point of AB and PQ.
Since , $d(PQ_{z,} ABz) + d\{(RI)z(ST)w, (ST)(ST)w\} = 0$
Then from (3.1.3) $d((AB)z, (ST)(ST)w) = 0$
yield that

$$(ST)(ST)w = (AB)z = (RI)(ST)w$$

or

$$(ST)(AB)z = (AB)z = (RI)(AB)z$$

Hence, $(AB)z = (ST)w$ is a common fixed point of ST and RI consequently, ABz is a common fixed point of AB, ST, PQ, and RT. The proof is similar when the pair (ST, RI) is assumed as compatible and reciprocal continuous.

Now if v is any common fixe point in ST and RI,
Then $d(ABz, PQz) + d(STv, RIv) = 0$
And so by (3.1.3), we have

$$d(ABz, STv) = 0 \quad \text{or} \quad d(ABz, v) = 0$$

yield that $ABz = v$

Hence, ABz is the unique common fixed point of (ST, RI). Consequently, on switching the role of pair (AB, PQ) and (ST, RI) as above, it can be seen that ABz is the unique common fixed point of (AB, PQ).
Again using (3.1.2) we have

$$F\{d(AB(AB)z,STx_{2n+1}), d(PQ, (AB)z, RIx_{2n+1}), d(PQ(AB)z, AB(AB)z)),$$
$$d(RIx_{2n+1}, STx_{2n+1}), d(PQ(AB)z, STx_{2n+1}), d(RIx_{2n+1}, AB(AB)z)\} \le 0$$

Or

$$F\{d(ABz,STx_{2n+1}), d((AB)z, RIx_{2n+1}), d((ABz),(ABz)), d(RIx_{2n+1},STx_{2n+1}),$$
$$d((ABz), STx_{2n+1}), d(RIx_{2n+1}, ABz)\} \le 0.$$

Letting $n \longrightarrow \infty$, we get

$$F(d\{ABz,z), d(ABz,z), 0, 0, d(ABz,z), d(z, ABz)\} \le 0$$

contradicting (F_3), thus $ABz = z$.
Hence, z is the unique common fixed point of AB, ST, PQ, RI.
Similarly this result remains true if we consider (3.1.4)(b) instead of (3.1.4)(a).
Now by (3.1.5), we have

$$Az = A(ABz) = A(BAz) = (AB)Az\ ;\ Az = A(PQz) = (PQ)Az$$

And

$$Bz = B(ABz) = (BA)Bz = (AB)Bz\ ;\ Bz = B(PQz) = (PQ)Bz$$

It follows that AZ and Bz are the common fixed point of (AB, PQ). But since z is the unique common fixed point of (AB, PQ), we have

$z = Az = Bz = ABz = PQz.$

Similarly, it can be proof that $z = Sz = Tz = STz = RIz$

Hence, z is the unique common fixed point of A, B, S, T, PQ, and RI.

Further from (3.1.6), we have

$$Pz = P(ABz) = (AB)Pz \; ; \; Pz = P(PQz) = P(QPz) = (PQ)Pz.$$
$$Qz = Q(ABz) = (AB)Pz \; ; \; Qz = Q(PQz) = (QP)Qz = (PQ)Qz.$$

Similarly, $Rz = R(ABz) = (AB)Rz \; ; \; Rz = R(PQz) = (PQ)Rz.$

and $Iz = I(ABz) = (AB)Iz \; ; \; Iz = I(PQz) = (PQ)Iz.$

Hence, Pz, Qz are the common fixed points of (AB, PQ) but by the uniqueness of z, $z = Pz = Qz$

Similarly, Rz, Iz are the common fixed points of (AB, PQ) implies that $z = Rz = Iz$.

Hence, z is the unique common fixed point of A, B, R, S ,T, P, Q and I.

Case II: Suppose that $d(PQx_{2n}, ABx_{2n}) + d(RIx_{2n+1}, STx_{2n+1}) = 0$.

Then, $ABx_{2n} = PQx_{2n}$ and $RIx_{2n+1} = STx_{2n+1}$

implies that v_1, $w_1$such, that

$$v_1 = ABw_1 = PQw_1.$$

Similarly there exist v_2 , w_2 such that $v_2 = STw_2 = RIw_2$.

Since, $d(ABw_1, PQw_1) + d(STw_2, RIw_2) = 0$,

then by (3.1.3), it follows that

$d(ABw_1, STw_2) = 0$ implies that $v_1 = ABw_1 = STw_2 = v_2$.

Since the pair (AB, PQ) is occasionally weakly compatible and $ABw_1 = PQw_1$, then

$PQv_1 = (PQ)(AB)w_1 = (AB)(PQ)w_1= (AB)v_1$. Similarly $STv_2 = RIv_2$.

Now we set $y_1 = ABv_1$, $y_2 = STv_2$.

Since, $d(ABv_1, PQv_1) + d(STv_2, RIv_2) = 0$,

Then from (3.1.3), $d(ABv_1, STv_2) = 0$ or $ABv_1 = STv_2$ or $y_1 = y_2$.

Thus, $ABv_1 = PQv_1 = STv_2 = RIv_2$.

But since $v_1 = v_2$,

We have $ABv_1 = PQv_1 = STv_1 = RIv_1$

i.e., v_1 is the coincidence point of AB, PQ, ST, and RI.

Again set $w = ABv_1$ then by $ABv_1 = PQv_1$ and weak compatibility of (AB, PQ), it follows that

$(AB)w = (AB)(ABv_1) = (AB)(PQv_1) = (PQ)(ABv_1) = (PQ)w.$

Thus, w is the coincidence point of AB and PQ. Also weak compatibility of (ST, RI) follows that $(ST)w = (ST)(RI)v_1 = (RI)(ST)v_1 = (RI)w$.

Hence w is a common coincidence point of AB, PQ, ST, and RI.

Since, $d(ABw, PQw) + d(STw, RIw) = 0$ then from (3.1.3), it follows that

$$d(ABw, STw) = 0,$$

implies that $ABw = STw$.

Thus , $ABw = STw = PQw = RIw$

On the other hand $d(ABw, PQw) + d(STv_1, RIv_1) = 0$ then from (3.1.3), we have $d(ABw, STv_1) = 0$ implies that $ABw = STv_1$,

Therefore, $w = ABw = STw = PQw = RIw$

Hence w is common fixed point of AB, PQ, ST and RI.

If $d(PQx_{2n}, ABx_{2n}) + d(RIx_{2n-1}, STx_{2n-1}) = 0$, for some n, the claim that there exist a common fixed point of AB, ST, PQ, RI can be proved similarly.

Rest of the proof is identical to case (1).

Now we finish the example to demonstrate the validity of the hypothesis and the degree of generality of our result.

Example 3.3. Let X = [0, ∞) be endowed with the usual metric d. Define

$Ax = \begin{cases} x, & x \in [0,2) \\ 1, & x \in [2,\infty) \end{cases}$, $Bx = \begin{cases} 1, x \in [0,2) \\ \frac{1}{\sqrt{x}}, \in [2,\infty) \end{cases}$

$Rx = \begin{cases} 2\sqrt{x}, & x \in [0,1) \\ 2x^4 - 1, & x \in [1,\infty) \end{cases}$, $Ix = \begin{cases} x, x \in [0,1) \\ \sqrt{x}, x \in [1.\infty) \end{cases}$,

$Sx = \begin{cases} 1, x \in [0,1) \\ \frac{1}{x}, x \in [1,\infty) \end{cases}$ $Tx = \begin{cases} 1, x \in [0,1) \\ x, \in [1,\infty) \end{cases}$,

$Px = \begin{cases} \sqrt{x}, x \in [0,1) \\ x, x \in [1,\infty) \end{cases}$, $Qx = \begin{cases} x, x \in [0,1\} \\ 2\sqrt{x} - 1, x \in [1,\infty) \end{cases}$,

Then it can easly verified that

(3.3.1) AB(X)⊆RI(X) and ST(X) ⊆ PQ(X)

(3.3.2) Define $F : R_+^6 \to R$ by $F(t_1, t_2, t_3,t_4,t_5, t_6) = t_1^2 - \frac{1}{4}t_2^2$ then $F \in \bar{F}$ and
F(d(ABx,STy),d(PQx,RIy),d(PQx,ABx),d(RIy,STy),d(PQx,STy),d(RIy,ABx))
$=[d(ABx,STy)]^2 - \frac{1}{4}[d(PQx,RIy)]^2 = |ABx - STy|^2 - |PQx - RIy|^2 \leq 0 \forall$ x, y ∈ X.

(3.3.4): there exists a sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ in X
such that $x_n = \{1 - \frac{1}{n}\} \to 1$, $ABx_n \to 1$ and $PQx_n = \sqrt{x_n} \to 1$ as $n \to \infty$.
Also $(AB)(PQ)x_n = (AB)\sqrt{x_n} \to 1$ and $(PQ)(AB)x_n = (PQ)1 \to 1$.
Therefore, $\lim_{n\to\infty} d\{(AB)(PQ)x_n, (PQ)(AB)x_n\} = 0$,
i.e., the pair (AB, PQ) is compatible and continuous maps.
Also here 1 and $\frac{1}{4}$ are the coincidence points of the pair (RI, ST) and we have
$(RI)(ST)_1 = (RI)_1 = 1 = (ST)(RI)_1$; $(RI)(ST)_{1/4} = (RI)_1 = 1 = (ST)_1 = (ST)(RI)_{1/4}$,
i.e. (RI, ST) is the pair of occasionallyweakly compatible mappings

(3.1.5) the pair (A, B), (A, PQ), (B, PQ), (S, T), (S, RI) and (T, RI) commute at the common fixed point 1 of AB, ST, PQ and RI.

(3.1.6) the pair (P, Q), (P, AB), (Q, AB), (R, AB), (R, PQ), (I, AB) and (I, PQ) commute at 1.

Thus, all the conditions of the theorem 3.1 are satisfied and 1 is the unique common fixed point of A, B, S, T, P, Q, R and I.

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